PARTITIONING THE PRODUCTION PROCESSES WHEN ASSEMBLING A BORE AND SHAFT

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Partitioning the production processes when assembling a bore and shaft

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ABSTRACT: The problem of assembling of two mating parts is considered. Making use of the probability distributions of the bore and shaft, and the specifications on the fit between them, it is desired to define the dimensions of separate groups for the bore and for the shaft that these parts can be placed in, such that the specifications on the fit when parts in these groups are assembled will be met with certainty. This is accomplished by formulating an optimization problem such that the number of such groups is made a minimum subject to a specified lower bound on the sum of the probabilities of a sample lying in each group, and introducing a method of generating approximate solutions. The method makes use of the 'solver' and 'chart wizard' functions that are contained in Microsoft Excel©. Using the dimensions of the groups, an expression for the expected number of assemblies generated by each group is presented.

Keywords: Assembly; Design for assembly; Inspection planning; Precision assembly; Quality assurance; Tolerance analysis; Truncated probability density function.

1. Introduction

The problem to be considered concerns the assembly of two mating parts. A survey can be found in Chase and Parkinson, 1991. The problem of mechanical assembly is discussed in Nof, Wilhelm and Warnecke, 1997 and Whitney, 2004. In what follows, it is assumed that manufacturing processes are in place for producing two mating parts, which will be generically identified as a bore and a shaft. Making use of the probability distributions of the bore and shaft, and the specifications on the fit between them, it is desired to define the dimensions of separate groups for the bore and for the shaft that these parts can be placed in, such that the specifications on the fit when parts in these groups are assembled will be met with certainty. This will be accomplished by formulating an optimization problem such that the number of such groups is made a minimum subject to a specified lower bound on the probability that the assembled items satisfy the desired conditions on the fit, and introducing a method of generating approximate solutions. Using the dimensions of the groups, an expression for the expected number of assemblies generated by each group will be presented.

Nomenclature

b_L	minimum value for a bore's group after inspection,
b_U	maximum value for a bore's group after inspection,
$B\{k, m, p\}$	binomial probability mass function,
$F_B\{k, m, p\}$	cumulative distribution function for a binomial probability mass function,
f_X	probability density function of the bore X,
f_Y	probability density function of the shaft Y,
m	number of items produced when $n^{(x)} = n^{(y)}$,
MTR	modified tolerance rectangle,
n	number of TRs used in the partitioning process,
$n^{(x)}$	number of bores produced in a production run,
$n^{(y)}$	number of shafts produced in a production run,
$N_i^{(x)}$	random variable equal to the number of bores of those produced that are in the i^{th} interval of the x axis,
$N_j^{(y)}$	random variable equal to the number of shafts of those produced that are in the j^{th} interval of the y axis,
$p_i^{(x)}$	probability that a bore lies in the i^{th} interval of the x axis,
$p_j^{(y)}$	probability that a shaft lies in the j^{th} interval of the y axis,
P_A	probability of acceptance of the bore and the probability of acceptance of the shaft occurring in a TR or MTR,
P_M	probability of a sample (X, Y) occurring in a TR or MTR,
s_L	minimum value for a shaft's group after inspection,
s_U	maximum value for a shaft's group after inspection,
TR	tolerance rectangle,
t_X	tolerance of the bore's group,
t_Y	tolerance of the shaft's group,

U_r	number of successful assemblies generated in the r[th] MTR,
w_{MAX}	maximum possible value for the fit *W*,
$w_{MAX}^{(d)}$	design value for the maximum for the fit *W*,
w_{MIN}	minimum possible value for the fit *W*,
$w_{MIN}^{(d)}$	design value for the minimum for the fit *W*,
W	random variable that represents the difference between bore and shaft; the fit,
X	random variable that represents the diameter of the bore before inspection,
Y	random variable that represents the diameter of the shaft before inspection,
Δw	tolerance for the fit,
$\Delta w^{(d)}$	design tolerance for the fit,
ε_X	lower limit of the pdf for the bore,
ε_Y	lower limit of the pdf for the shaft,
γ_X	range of the pdf for the bore,
γ_Y	range of the pdf for the shaft,
μ_X	parameter for the expected value of the normal distribution of the bore,
μ_Y	parameter for the expected value of the normal distribution of the shaft,
σ_X	parameter for the standard deviation of the normal distribution of the bore,
σ_Y	parameter for the standard deviation of the normal distribution of the shaft,
Σ	sum of probabilities to be maximized,
$\{[b_L, s_L], t_X, t_Y\}$	rectangle with origin $[b_L, s_L]$ and sides t_X and t_Y.

2. Problem formulation

It is assumed that manufacturing processes are in place for producing a bore and a shaft. The fit between the bore and the shaft is defined as the difference in the dimensions of the diameters of the bore and of the shaft. In the assembly problem as defined in this paper, it is desired to fit the bore and shaft together in such a manner that the event that the specification for the maximum tolerance of the fit is not exceeded has a probability equal to one. Additionally, a constraint that either the event that the specification for the maximum fit is exceeded or the event that the specification for the minimum fit not be exceeded have a probability equal to zero can also be applied. This is possible due to the fact that the tolerance of the fit is defined as the difference between the maximum and minimum fits.

It is assumed that many bores and shafts will be produced by two statistically independent processes, and it is desired that they should be paired such that, if possible, the above conditions on the fit are satisfied. To that end, each of the items produced by each process will be divided into groups, and it is desired that the number of such groups be made a minimum subject to a specified lower bound on the probability that the assembled items satisfy the desired conditions on the fit. One advantage of forming such groups is that go/no-go gauges can be used. This paper presents a method for computing the dimensions associated with each group.

If the manufacturing processes are in statistical control, the values associated with various parameters can be estimated. Let X represent a continuous random variable that denotes the diameter of the bore, and Y represent a continuous random variable that denotes the diameter of the shaft. It is assumed that X and Y are independent random variables. The fit will be denoted by W, where $W = X - Y$. Let the probability density function (pdf) of the bore be distributed between $\varepsilon_X > 0$ and $\varepsilon_X + \gamma_X$, $\gamma_X > 0$, and the pdf of the shaft be distributed between $\varepsilon_Y > 0$ and $\varepsilon_Y + \gamma_Y$, $\gamma_Y > 0$. It follows that the largest and smallest possible values for the fit are given by $w_{MAX} = \varepsilon_X + \gamma_X - \varepsilon_Y$ and $w_{MIN} = \varepsilon_X - \varepsilon_Y - \gamma_Y$, and the difference between the two is equal to $\gamma_X + \gamma_Y$. For design purposes, the maximum fit between bore and shaft is denoted by $w_{MAX}^{(d)}$ and minimum fit is denoted by $w_{MIN}^{(d)}$. Let the design value for the tolerance of the fit be denoted as $\Delta w^{(d)}$ where

$$\Delta w^{(d)} = w_{MAX}^{(d)} - w_{MIN}^{(d)} . \quad (1)$$

It can be seen from (1) that since $\Delta w^{(d)}$ depends on the difference between $w_{MAX}^{(d)}$ and $w_{MIN}^{(d)}$, many different configurations can be found without violating the condition that the specification for the maximum tolerance is not exceeded with a probability equal to one.

If $\Delta w^{(d)} < \gamma_X + \gamma_Y$, the specification cannot be satisfied with certainty. One action that can be taken is to inspect the processes in order to place the samples into groups. The effect of the inspection process may be to truncate the pdfs of the bore and of the shaft that belong to a particular group. Each truncated pdf of a bore's group will be then be distributed between its smallest value b_L and its largest value $b_U > b_L$, where the group's tolerance is defined by

$$t_X = b_U - b_L, \tag{2}$$

and each truncated pdf of a shaft's group will be distributed between its smallest value s_L and its largest value $s_U > s_L$, where the group's tolerance is defined by

$$t_Y = s_U - s_L, \tag{3}$$

where $b_L \geq \varepsilon_X$, $b_U \leq \varepsilon_X + \gamma_X$, $s_L \geq \varepsilon_Y$ and $s_U \leq \varepsilon_Y + \gamma_Y$.

One possible design is to specify that the probabilities that $\{W > w_{MAX}^{(d)}\}$ and $\{W < w_{MIN}^{(d)}\}$ both be equal to zero. It then follows that

$$w_{MAX}^{(d)} = b_U - s_L = b_L + t_X - s_L, \tag{4}$$

$$w_{MIN}^{(d)} = b_L - s_U = b_L - t_Y - s_L, \tag{5}$$

and the tolerance of the fit is then given by

$$\Delta w^{(d)} = t_X + t_Y. \tag{6}$$

In this case,

$$P\left\{W \leq \Delta w^{(d)}\right\} = 1. \tag{7}$$

For the procedure to be discussed, it is desired to fit the bore and shaft in such a manner that (7) is valid, but the values of the minimum and maximum fit are allowed to differ for different groups. Under these conditions, two possible approaches to choosing the groups will be discussed. One approach is purely geometric, and does not use the expressions for the pdfs, but does use their ranges. The other approach makes use of the pdfs.

3. Geometric approach to the assignment of truncation values when no constraints are present

Using a two-dimensional Cartesian plot, let the dimension of the bore be plotted on the *x* axis, and the dimension of the shaft be plotted on the *y* axis. The coordinates ($\varepsilon_X, \varepsilon_Y$), ($\varepsilon_X + \gamma_X, \varepsilon_Y$), ($\varepsilon_X, \varepsilon_Y + \gamma_Y$) and ($\varepsilon_X + \gamma_X, \varepsilon_Y + \gamma_Y$) define the four corners of a rectangle that represents the sample description space (SDS) for (*X,Y*). Using an inspection process, it is desired to partition the sample description space in such a way that, if possible, (6) is satisfied for each partition. Each partition is defined by a rectangle, to be called a tolerance rectangle (TR), with corners (b_L, s_L), (b_U, s_L), (b_L, s_U) and (b_U, s_U). The corner (b_L, s_L) will be called the origin of the TR, and, as will be illustrated below, the location of this origin is not necessarily unique. A design value for the fit $\Delta w^{(d)}$ is assumed to be given, values for t_X and t_Y must be found, and then b_U and s_U can be found from (2) and (3), thus uniquely defining a TR. A TR can be specified by using the notation $\{[b_L, s_L], t_X, t_Y\}$. In this section, the TRs will be chosen without making use of the expressions for the pdfs of the bore and shaft. The objective is to generate the minimum number of disjoint partitions that can be contained in the SDS, and simultaneously have their union be exhaustive. Since it is desired to fill the sample space with the minimum number of TRs, the area of each TR should be chosen as large as possible, subject to (6). This can be accomplished, if constraints are not violated, by setting

$$t_X = t_Y = \Delta w^{(d)}/2 . \qquad (8)$$

Consider the situation where $\gamma_X \geq \gamma_Y > 0$.

Case a).

If

$$\gamma_X \geq \Delta w^{(d)}/2 \qquad (9)$$

and

$$\gamma_Y \geq \Delta w^{(d)}/2 \qquad (10)$$

then use (8) for the tolerances on the bore and shaft.

Case b).

If (9) applies, but

$$\gamma_Y < \Delta w^{(d)}/2 \qquad (11)$$

then use

$$t_Y = \gamma_Y \,. \qquad (12)$$

and

$$t_X = \Delta w^{(d)} - \gamma_Y \,. \qquad (13)$$

Case c).

If

$$\gamma_X < \Delta w^{(d)}/2 \qquad (14)$$

and

$$\gamma_Y < \Delta w^{(d)}/2 \qquad (15)$$

then

$$t_X = \gamma_X \,, \qquad (16)$$

$$t_Y = \gamma_Y \qquad (17)$$

and

$$\Delta w = t_X + t_Y < \Delta w^{(d)} \,. \qquad (18)$$

If $\gamma_Y \geq \gamma_X$, then equations (9) to (18) are valid if the subscripts *X* and *Y* are interchanged. These equations may be applied as many times as necessary in order to find a set of mutually exclusive and exhaustive tolerance rectangles , which will be denoted as TR_i, *i* = 1, 2, …., *n*. There follows two examples of the application of the above equations.

3A. An example with no constraints on the minimum or maximum fit

Let ε_X =0.65, γ_X = 3.15, ε_Y = 1.00, γ_Y =1.98 and $\Delta w^{(d)}$ = 2.0. Since (9) and (10) are valid, (8) yields $t_X = t_Y = 1.0$. It is possible to fit three TRs with $t_X = t_Y = 1.0$ into the SDS, as denoted in Table 1 as TR_1, TR_2 and TR_3 and as shown in Figure 1, where use is made of ‘data labels’ for the corners of some of the TRs. When using the data labels, the *x* coordinate was placed above the *y* coordinate. There then exists an “unfilled” rectangle $\{[0.65, 2.00], 3.15, 0.98\}$. Since (9) and (11) are valid, (12) and (13) yield t_Y = 0.98 and

$t_X =$ 1.02. It is then possible to fit TR_4, TR_5 and TR_6 in the SDS by placing them above TR_1, TR_2 and TR_3, respectively . Since intervals on the *x* axis must be disjoint, one possibility is to have intervals of width 0.02, 0.04 and 0.06 on the *x* axis in addition to the three intervals of width 1.00. However, this would lead to a greater number of TRs, so $t_X =$ 1.00 is also used for TR_4, TR_5 and TR_6 .There then remains an unfilled rectangle $\{[3.65, 1.00], 0.15, 1.98\}$, and two remaining TRs are shown in Table 1. The arrangement of TRs shown in Table 1 is not unique, but they do form an exhaustive set. The last column in Table 1 shows values for the ratio of $t_X\ t_Y$ to $\gamma_X\ \gamma_Y$, which represents the ratio of the area of the TR to the area of the SDS. Note that TR_4 is an interference fit, TR_i, $i =$ 1, 2, 5 and 6 are transition fits, and the rest of the TRs are clearance fits. The partition shown in Table 1 was arrived at using a 'one TR at-a-time' approach.

i	(b_L, s_L)	t_X	t_Y	w_{MIN}	w_{MAX}	Δw	$t_X\ t_Y / \gamma_X\ \gamma_Y$
1	(0.65, 1.00)	1.00	1.00	-1.35	0.65	2.00	0.16033
2	(1.65, 1.00)	1.00	1.00	-0.35	1.65	2.00	0.16033
3	(2.65, 1.00)	1.00	1.00	0.65	2.65	2.00	0.16033
4	(0.65, 2.00)	1.00	0.98	-2.33	-0.35	1.98	0.15713
5	(1.65, 2.00)	1.00	0.98	-1.33	0.65	1.98	0.15713
6	(2.65, 2.00)	1.00	0.98	-0.33	1.65	1.98	0.15713
7	(3.65, 1.00)	0.15	1.00	1.65	2.80	1.15	0.02405
8	(3.65, 2.00)	0.15	0.98	0.67	1.80	1.13	0.02357

Table 1. A list of TR_i, i = 1, 2, …, 8, for Example 3A, using only the constraint $\Delta w^{(d)} =$ 2.0.

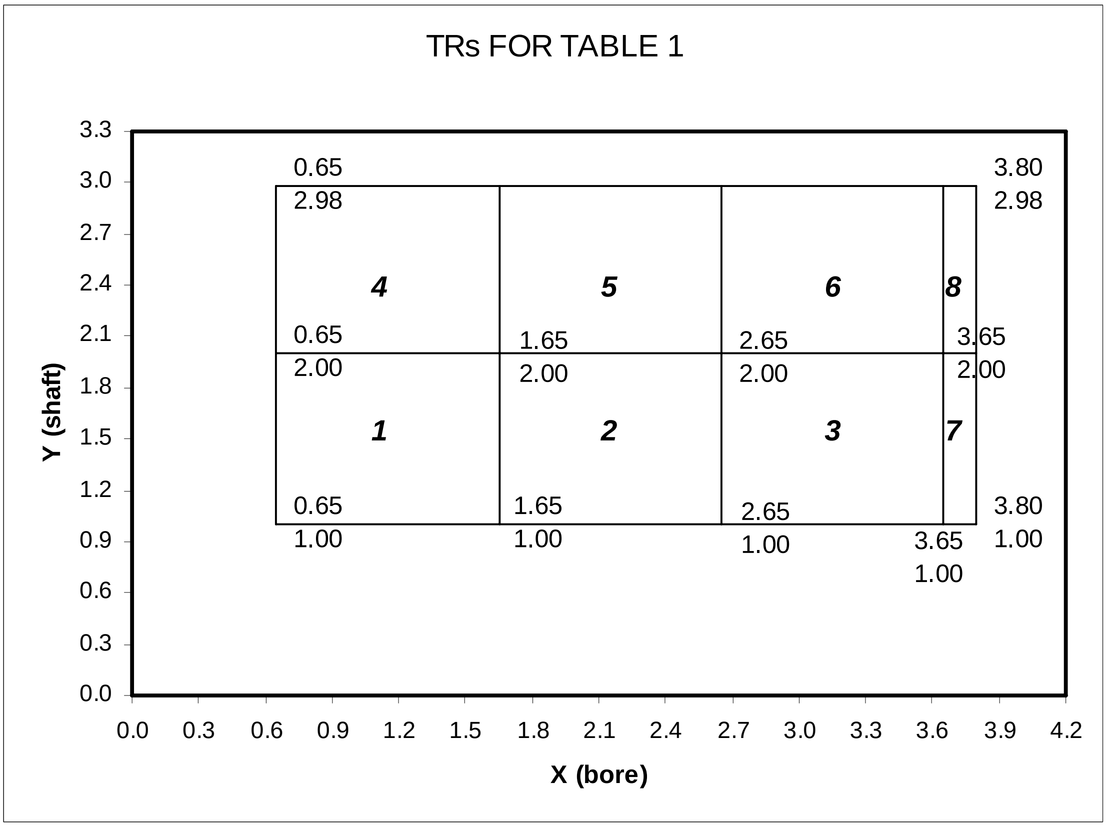

Figure 1. Eight TRs for Example 3A, using only the constraint $\Delta w^{(d)} = 2.0$.

4. Geometric approach to the assignment of truncation values when constraints are present

If the minimum fit were to be constrained to have a value greater than or equal to $w_{MIN}^{(d)}$ for every TR, the upper left hand corner of every TR would be bounded by the straight line given by

$$y = x - w_{MIN}^{(d)} \quad . \tag{19}$$

If the maximum fit were to be constrained to have a value less than or equal to $w_{MAX}^{(d)}$ for every TR, the lower right hand corner of every TR would be bounded by the straight line given by

$$y = x - w_{MAX}^{(d)} \quad . \tag{20}$$

4A. An example, with constraints on the minimum and maximum fits

Assume that in example 3A that $w_{MIN}^{(d)} = 0$ and $w_{MAX}^{(d)} = 2.0 = \Delta w^{(d)}$. A 'one TR at-a-time' approach will be used. Applying (8), it is possible to find a TR with two corners that intersect the lines (constraining boundaries) defined by (19) and (20). This TR is denoted in Table 2 as TR_1 and is shown in Figure 2. It is then possible to find one other TR with two corners that intersect the constraining boundaries. This TR is shown in Table 2 as TR_2. There remains one trapezoidal region and three triangular regions between the two constraining boundaries that are not covered by TR_1 and TR_2. Table 2 shows four more TRs that have been constructed, which in turn leaves seven more uncovered triangular regions. Each of these triangular regions can have (smaller) TRs inserted, and this process can be continued until it is decided to ignore the remaining uncovered triangular regions. Using only the six TRs shown in Table 2, the ratio of the sum of the areas of these six TRs, which is equal to 2.674, divided by the area between the two constraining boundaries defined by (19) and (20), which is equal to 3.262, equals 0.820. The boundaries of the six TRs are shown in Figure 2 by making use of solid lines.

i	(b_L, s_L)	t_X	t_Y	w_{MIN}	w_{MAX}	Δw	$t_X\ t_Y$
1	(2.80, 1.80)	1.00	1.00	0.00	2.00	2.00	1.000
2	(1.80, 1.00)	1.20	0.80	0.00	2.00	2.00	0.960
3	(3.00, 2.80)	0.80	0.18	0.02	1.00	0.98	0.144
4	(2.30, 1.80)	0.50	0.50	0.00	1.00	1.00	0.250
5	(1.40, 1.00)	0.40	0.40	0.00	0.80	0.80	0.160
6	(3.00, 1.40)	0.40	0.40	1.20	2.00	0.80	0.160

Table 2. A list of six TRs for Example 4A, using the constraints $w_{MIN}^{(d)} = 0.0$ and $w_{MAX}^{(d)} = 2.0 = \Delta w^{(d)}$.

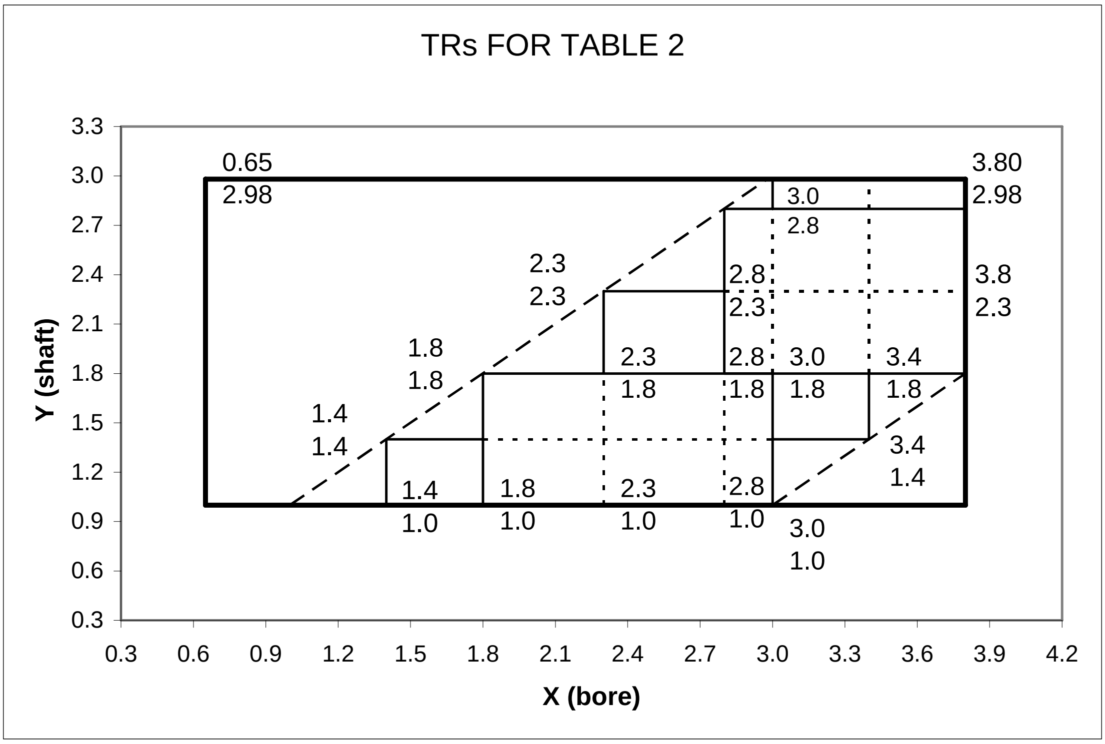

Figure 2. Six TRs for Example 4A, using the constraints $w_{MIN}^{(d)}$ = 0.0 and $w_{MAX}^{(d)}$ = 2.0 = $\Delta w^{(d)}$. The dashed lines define the MTRs.

In order to form disjoint intervals for *X* for the partitioning problem, it will be necessary to have six intervals [1.4, 1.8], (1.8, 2.3], (2.3, 2.8], (2.8, 3.0], (3.0, 3.4] and (3.4, 3.8]. The five intervals for *Y* are [1.0, 1.4], (1.4, 1.8], (1.8, 2.3], (2.3, 2.8], and (2.80, 2.98], yielding total of 17 'modified' TRs (MTRs), as shown in Figure 2. In Figure 2, dashed lines are used to show the boundaries of the MTRs. Once again, the partition that has been found is not unique. Inspection of Figure 2 shows that if the boundaries of TR_1 are moved by changing the value of b_U from 2.8 to 3.0, the number of MTRs can be reduced to 11. The new partition is shown in Figure 3 and in Table 3. The sum of the areas of these 11 MTRs is equal to 2.274.

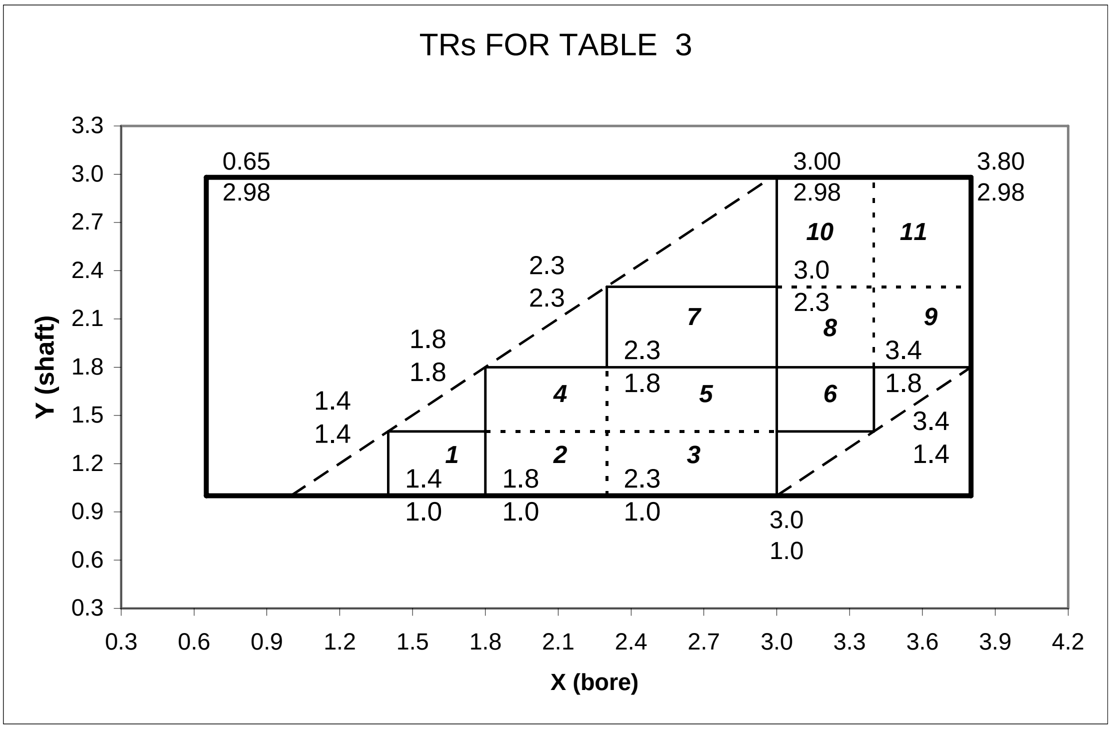

Figure 3. Five different TRs for Example 4A, using the constraints $w_{MIN}^{(d)} = 0.0$ and $w_{MAX}^{(d)} = 2.0 = \Delta w^{(d)}$. The dashed lines define the MTRs.

In order to form disjoint intervals for *X* for the partitioning problem, it will be necessary to have five intervals [1.4, 1.8], (1.8, 2.3], (2.3, 3.0], (3.0, 3.4] and (3.4, 3.8]. The four intervals for *Y* are [1.0, 1.4], (1.4, 1.8], (1.8, 2.3] and (2.30, 2.98], yielding total of eleven MTRs, as shown in Figure 3. Any sample for *X* with a value less than 1.0 will not yield an assembly that will be within specifications. However, if a sample for *X* lies in the interval [1.0, 1.4], and if a sample for *Y* lies in the interval [1.0, 1.4], there is a possibility that the sample will have a fit that will be within specifications. Similarly, if a sample for *X* lies in the interval (3.0, 3.4], and if a sample for *Y* lies in the interval [1.00, 1.40], there is a possibility that the fit will be within specifications. However, if a sample for *X* lies in the interval (3.4, 3.8], and if a sample for *Y* lies in the interval [1.00, 1.40], it will be necessary to have a constraint that states that such a combination will not be allowed. Similar considerations exist for other intervals that are adjacent to the constraining boundaries. In the next section, the probabilistic approach to defining the partitions will be used.

i	(b_L, s_L)	t_X	t_Y	w_{MIN}	w_{MAX}	Δw	$t_X\ t_Y$
1	(1.40, 1.00)	0.40	0.40	0.00	0.80	0.80	0.160
2	(1.80, 1.00)	0.50	0.40	0.40	1.30	0.90	0.200
3	(2.30, 1.00)	0.70	0.40	0.90	2.00	1.10	0.280
4	(1.80, 1.40)	0.50	0.40	0.00	0.90	0.90	0.200
5	(2.30, 1.40)	0.50	0.40	0.50	1.40	1.90	0.200
6	(3.00, 1.40)	0.40	0.40	1.20	2.00	0.80	0.160
7	(2.30, 1.80)	0.50	0.50	0.00	1.00	1.00	0.250
8	(3.00, 1.80)	0.40	0.50	0.70	1.60	0.90	0.200
9	(3.40, 1.80)	0.40	0.50	1.10	2.00	0.90	0.200
10	(3.00, 2.30)	0.40	0.58	0.12	1.10	0.98	0.232
11	(3.40, 2.30)	0.40	0.58	0.52	1.50	0.98	0.232

Table 3. A different list of MTRs for Example 4A, using the constraints $w_{MIN}^{(d)} = 0.0$ and $w_{MAX}^{(d)} = 2.0 = \Delta w^{(d)}$.

5. A probabilistic approach to the computation of truncation values

The following concept has been used by Sweet and Tu, 2007. Let the sum of the probability of acceptance of the bore and the probability of acceptance of the shaft be given by

$$P_A = \int_{b_L}^{b_U} f_X(x)dx + \int_{s_L}^{s_U} f_Y(y)dy \qquad (21)$$

where f_X and f_Y are the (known) pdfs for the bore and shaft before inspection. It is desired to compute values for b_L, b_U, s_L and s_U that will maximize the sum of the probability of acceptance of bore and shaft, subject to the constraint given by (6). The resulting solution will yield a TR. Another approach is to compute values for b_L, b_U, s_L and s_U that will maximize the probability of a sample being in a TR, given by

$$P_M = \int_{b_L}^{b_U} f_X(x)dx \int_{s_L}^{s_U} f_Y(y)dy \ , \qquad (22)$$

subject to the constraint given by (6). In Sweet & Tu, 2007 it was shown that the optimization could be accomplished using the "solver" function in Microsoft Excel©. The function to be maximized is given by (21) or by (22). In order to solve the partitioning problem, a number of TRs, denoted by *n*, will be used instead of using the one-at-a-time approach. The constraints for each TR_i are given by

$$b_{L,i} \geq \varepsilon_X \text{ ,} \tag{23}$$

$$b_{U,i} \leq \varepsilon_X + \gamma_X \text{ ,} \tag{24}$$

$$b_{U,i} \geq b_{L,i} \text{ ,} \tag{25}$$

$$s_{L,i} \geq \varepsilon_Y \text{ ,} \tag{26}$$

$$s_{U,i} \leq \varepsilon_Y + \gamma_Y \text{ ,} \tag{27}$$

and

$$s_{U,i} \geq s_{L,i} \tag{28}$$

for i = 1, 2, …, n. If $w_{MIN}^{(d)}$ is also specified, then the constraint

$$b_{L,i} - s_{U,i} \geq w_{MIN}^{(d)} \tag{29}$$

is also used, and if $w_{MAX}^{(d)}$ is also specified, then the constraint

$$b_{U,i} - s_{L,i} \leq w_{MAX}^{(d)} \tag{30}$$

is also used.

Let

$$\Sigma_A = \sum_{i=1}^{n} P_{A,i} \tag{31}$$

and

$$\Sigma_M = \sum_{i=1}^{n} P_{M,i} \text{ .} \tag{32}$$

The approach to be used is to apply solver in order to find TR_1, using the above constraints. The constraints in (23) to (28) can be written in a more concise form, but they have been expressed in a form for use directly in solver. By searching all of the regions adjacent to TR_1, a combination of a new TR_1 and a TR_2 can be found. This can be done by adding a constraint such as $b_{L,2} = b_{U,1}$ or $s_{L,2} = s_{U,1}$ and using solver in order to maximize (31) or (32), with n = 2. If no constraining boundaries are present, this procedure can be continued until the allowable SDS is exhausted, or the probabilities Σ_A or Σ_M are at least equal to a specified lower bound. If constraining boundaries are present, this procedure can be continued until the probabilities Σ_A or Σ_M are at least equal to a specified lower bound. The procedure does not necessarily lead to an optimal solution, but will produce an approximate solution. The optimization problem appears to be a ‘set covering’ problem. The procedure will be illustrated with numerical examples when constraints are present.

5A. An example for uniformly distributed bore and shaft, with constraints present, using Σ_M .

The numerical values given in Example 4A will be used, and (32) will be applied. Using solver with n = 1, a TR is generated with $t_X = t_Y = 1.0$, but whose location is not unique. Letting n = 2, and trying both $b_{L,2} = b_{U,1}$ and $s_{L,2} = s_{U,1}$, the better solution is shown in Figure 4. Solver was then used for increasing values of n until n = 7 is reached. As n increased, the number of MTRs increased. A solution using n = 5 that yielded 11 MTRs is shown in Table 4 and in Figure 5. In the solution shown in Figure 5, the line with coordinates (2.98, 1.39) and (3.39, 1.39) was moved in a vertical direction until it coincided with the line with coordinates (2.98, 1.495) and (3.39, 1.495) in order to reduce the number of MTRs. The line with coordinates (3.39, 1.80) and (3.80, 1.80) was moved in a vertical direction until it coincided with the line with coordinates (3.39, 1.99) and (3.80, 1.99). This change yielded 11 MTRs. The last column of Table 4 shows the probability $P_{M,i}$ of a sample (X, Y) lying in MTR_i, i = 1, …, 11.

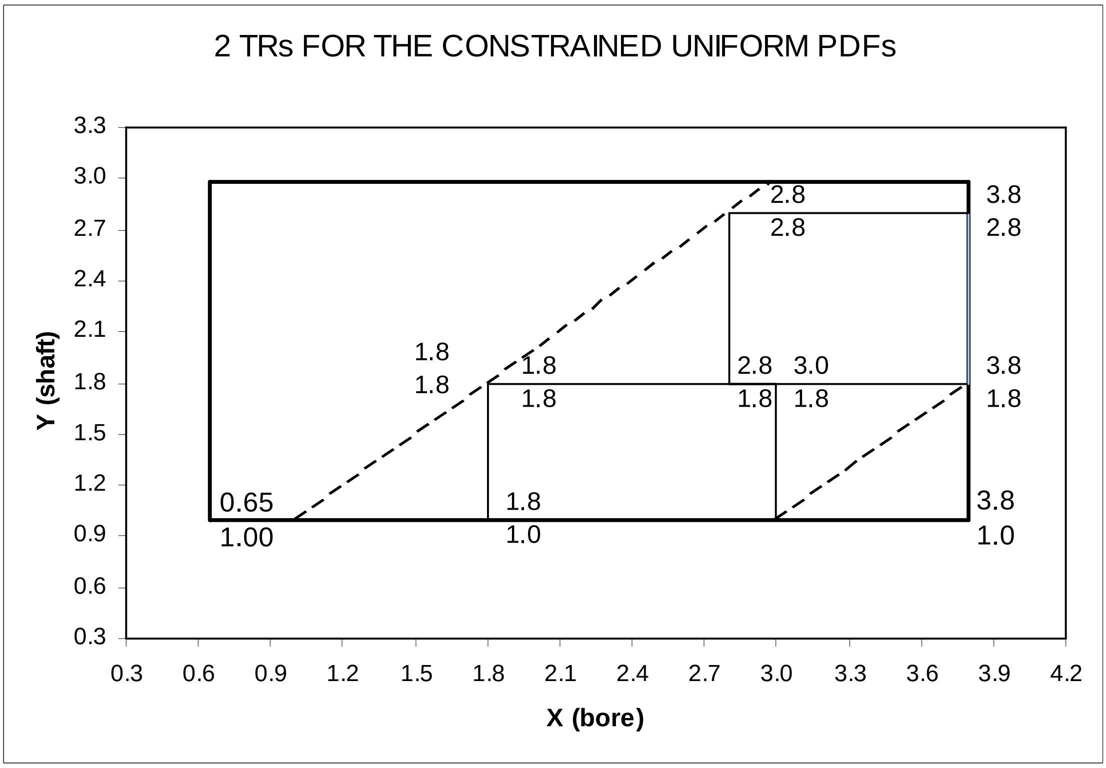

Figure 4. Two TRs for Example 5A, using uniform pdfs and the constraints $w_{MIN}^{(d)} = 0.0$ and $w_{MAX}^{(d)} = 2.0 = \Delta w^{(d)}$.

Summing these probabilities yields the probability of a sample (*X*, *Y*) lying in one of the 11 MTRs equal to 0.3984. The conditional probability of a sample (*X*, *Y*) lying in one of the 11 MTRs, given that the sample lies between the two constraining boundaries, is equal to 0.7790.

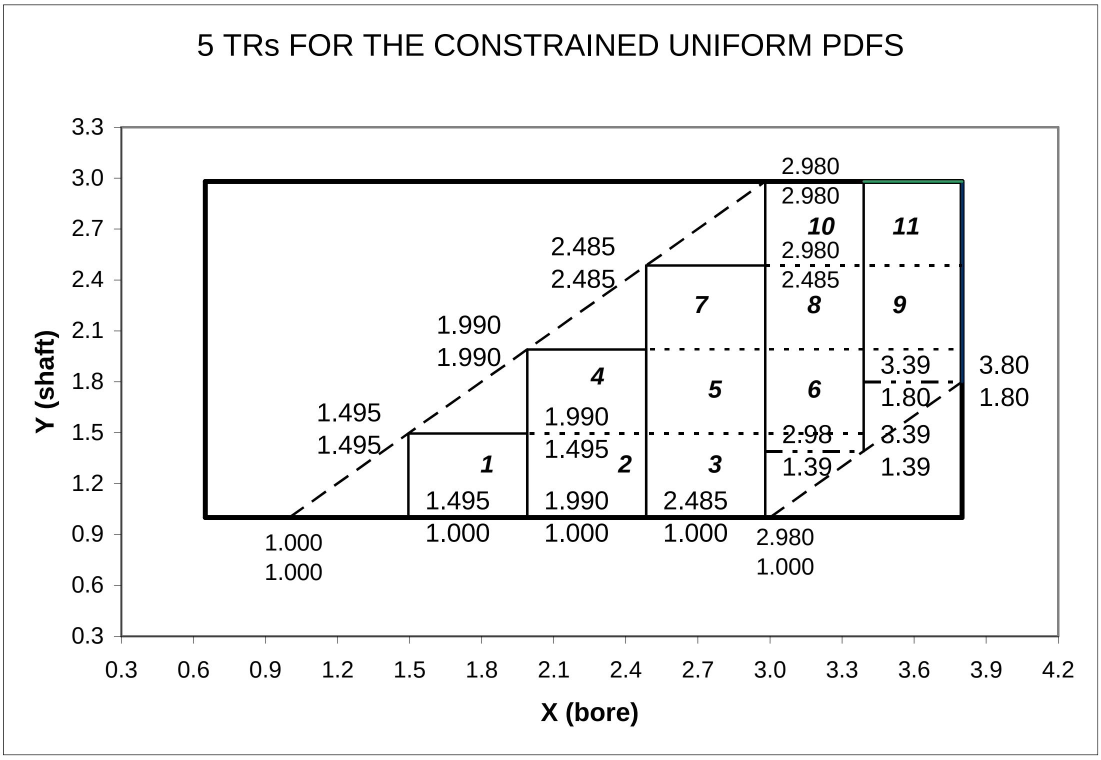

Figure 5. Five TRs for Example 5A, using uniform pdfs and the constraints $w_{MIN}^{(d)}$ = 0.0 and $w_{MAX}^{(d)}$ = 2.0 = $\Delta w^{(d)}$, and showing the MTRs.

i	(b_L, s_L)	t_X	t_Y	w_{MIN}	w_{MAX}	Δw	P_M
1	(1.495, 1.000)	0.495	0.495	0.000	0.990	0.990	0.03929
2	(1.990, 1.000)	0.495	0.495	0.495	1.485	0.990	0.03929
3	(2.485, 1.000)	0.495	0.495	0.990	1.980	0.990	0.03929
4	(1.990, 1.495)	0.495	0.495	0.000	0.990	0.990	0.03929
5	(2.485, 1.495)	0.495	0.495	0.495	1.485	0.990	0.03929
6	(2.980, 1.495)	0.410	0.495	0.990	1.895	0.905	0.03254
7	(2.485, 1.990)	0.495	0.495	0.000	0.990	0.990	0.03929
8	(2.980, 1.495)	0.410	0.495	0.990	1.895	0.905	0.03254
9	(3.390, 1.990)	0.410	0.495	0.905	1.810	0.905	0.03254
10	(2.980, 2.485)	0.410	0.495	0.000	0.905	0.905	0.03254
11	(3.390, 2.485)	0.410	0.495	0.410	1.315	0.905	0.03254

Table 4. Eleven MTRs for Example 5A, using uniform pdfs and the constraints $w_{MIN}^{(d)}$ = 0.0 and $w_{MAX}^{(d)}$ = 2.0 = $\Delta w^{(d)}$.

5B. An example for normally distributed bore and shaft, with constraints present, using Σ_M .

Using the same parameters for the SDS as in the previous examples, let *X* have a truncated normal distribution with parameters μ_X = 1.91 and σ_X = 0.63, and let *Y* have a truncated normal distribution with parameters μ_Y = 1.99 and σ_Y = 0.33. Thus, *Y* has a natural tolerance, but unlike the pdf for *Y*, the pdf for *X* is truncated in an unsymmetrical manner. Letting *n* = 1, the use of solver yielded TR_1 = $\{[2.129, 1.272], 1.144, 0.856\}$, which is pictured in Figure 6 as the rectangle with two corners that touch the constraining boundaries. In the next application of solver, *n* = 5 was used, and the solution was constrained by fixing the parameters of TR_1 and constraining each of the remaining four TRs to be adjacent to TR_1. It was found that two of the four TRs had much smaller values of $P_{M,i}$ than the other 3. Thus, *n* = 3 was used, yielding the results shown in Figure 6 and Table 5. It can be seen that the solution is simpler than in the case of the uniform pdfs because of the shape of the normal pdfs. In order to form disjoint intervals for *X* for the partitioning problem, it will be necessary to have four intervals [1.834, 2.129], (2.129, 2.417], (2.417, 3.272] and (3.272, 3.800]. The four intervals for *Y* are [1.000, 1.272], (1.272, 1.834], (1.834, 2.129] and (2.129, 2.417], yielding total of eight MTRs, as shown in Figure 6. Summing the probabilities in the last column of Table 5 yields the probability of a sample (*X*, *Y*) lying in one of the eight MTRs as equal to 0.34358. Using Sweet and Tu, 2006, the probability that $0 < W \leq 2$ = 0.46395, so the conditional probability of a sample (*X*, *Y*) lying in one of the eight MTRs, given that the sample lies between the two constraining boundaries, is equal to 0.74055. In order to increase this value, additional TRs are needed. For example, using *n* = 5 increases the conditional probability to 0.78787.

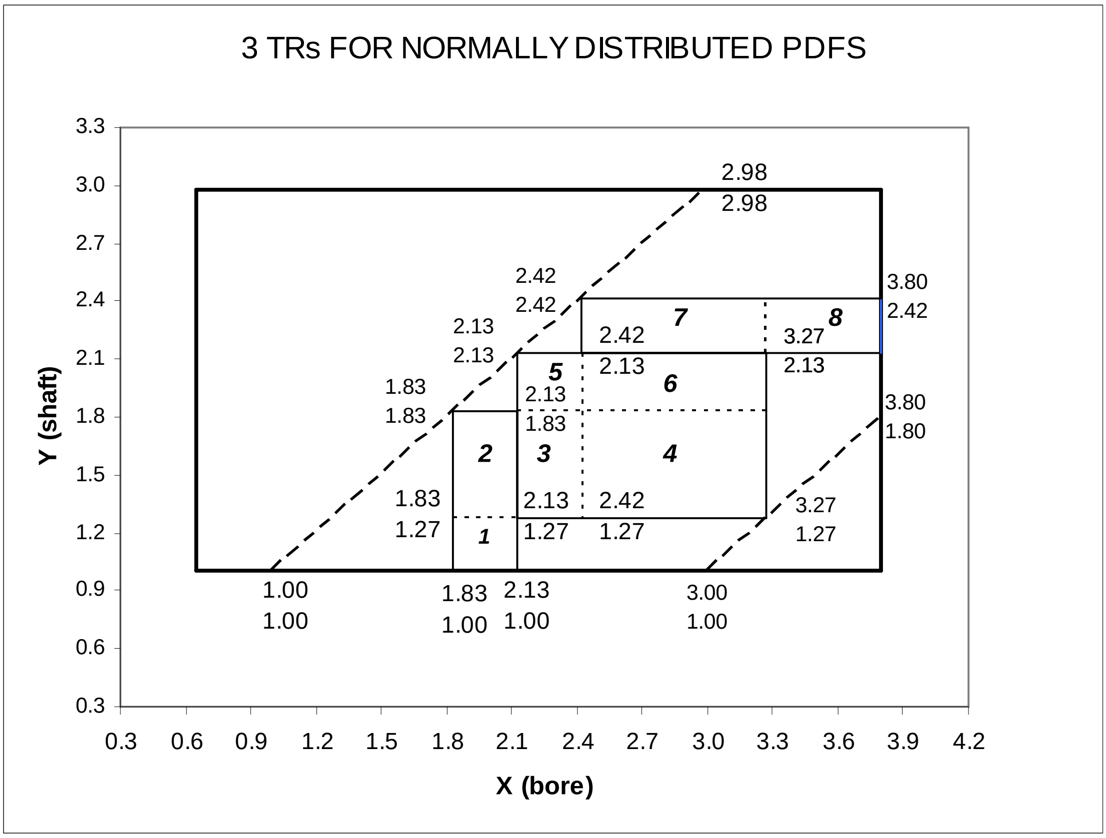

Figure 6. Three TRs for Example 5B, using normal pdfs and the constraints $w_{MIN}^{(d)}$ = 0.0 and $w_{MAX}^{(d)}$ = 2.0 = $\Delta w^{(d)}$, and showing the MTRs.

Any sample for X with a value less than 1.000 will not be within specifications. However, if a sample for X lies in the interval (1.000, 1.834), and if a sample for Y lies in the interval [1.000, 1.272], there is a probability equal to 0.0045 that the fit will be within specifications. Similarly, if a sample for X lies in the interval (2.129, 3.272), and if a sample for Y lies in the interval [1.000, 1.272], there is a probability equal to 0.0048 that the fit will be within specifications. However, if a sample for X lies in the interval (3.272, 3.800], and if a sample for Y lies in the interval [1.000, 1.834], it will be necessary to have a constraint that states that such a combination will not be allowed. Similar considerations exist for other intervals that are adjacent to the constraining boundaries. As can be seen from Table 5, the values for P_M for MTR_1 and MTR_8 are much smaller than those of the other MTRs and, if desired, can be ignored, resulting in the use of only six MTRs.

i	(b_L, s_L)	t_X	t_Y	w_{MIN}	w_{MAX}	Δw	P_M
1	(1.834, 1.000)	0.295	0.272	0.562	1.129	0.567	0.00254
2	(1.834, 1.272)	0.295	0.562	0.000	0.857	0.857	0.05734
3	(2.129, 1.272)	0.288	0.562	0.295	1.145	0.850	0.04788
4	(2.417, 1.272)	0.855	0.562	0.583	2.000	1.417	0.06084
5	(2.129, 1.834)	0.288	0.295	0.000	0.583	0.583	0.05444
6	(2.417, 1.834)	0.855	0.295	0.288	1.438	1.150	0.06918
7	(2.417, 2.129)	0.855	0.288	0.000	1.143	1.143	0.04792
8	(3.272, 2.129)	0.528	0.288	0.855	1.671	0.816	0.00343

Table 5. A list of eight MTRs for Example 5B, using normal pdfs and the constraints $w_{MIN}^{(d)} = 0.0$ and $w_{MAX}^{(d)} = 2.0 = \Delta w^{(d)}$.

6. The expected number of successful assemblies

After the groups are formed, it is of interest to evaluate their statistical properties. Let an MTR be defined by the i^{th} interval on the x axis and the j^{th} interval on the y axis. Assume that $n^{(x)}$ bores and $n^{(y)}$ shafts have been produced. Let

$$p_i^{(x)} = P\{X \text{ lies in the } i^{th} \text{ interval of the } x \text{ axis}\} \quad (33)$$

and

$$p_j^{(y)} = P\{Y \text{ lies in the } j^{th} \text{ interval of the } y \text{ axis}\}. \quad (34)$$

Let U_r equal the number of successful assemblies generated in MTR_r. Let the random variable $N_i^{(x)}$ equal the number of bores of the $n^{(x)}$ produced that are in the i^{th} interval on the x axis, and let the random variable $N_j^{(y)}$ equal the number of shafts of the $n^{(y)}$ produced that are in the j^{th} interval of the y axis. Then $N_i^{(x)}$ has a binomial probability mass function (pmf) with parameters $n^{(x)}$ and $p_i^{(x)}$, and $N_j^{(y)}$ has a binomial pmf with parameters $n^{(y)}$ and $p_j^{(y)}$. Let the binomial pmf be denoted by $B\{k, m, p\}$, where k is the number of successes in m independent and identical Bernoulli trials that each have a probability of success equal to p. Let $F_B\{k, m, p\}$ be the cumulative probability distribution function for the binomial pmf. The following assumption will be made in deriving an expression for the expected value of U_r. In order to illustrate why the assumption is used, assume that there are two groups on the x axis that are compatible with one group on the y axis, and assume that each of the two groups on the x axis each contain one bore. If the production process has contributed just one shaft that lies in the y

group, it can be used in just one of the two x groups, leaving the other one empty. In the derivation that follows, it is assumed that whatever group is being considered will be the one to receive all of the necessary parts if they are available. Thus, the expression to be derived can be considered to be an upper bound for the expected value of U_r. It follows that

$$P\{U_r = k \mid n^{(x)}, p_i^{(x)}, n^{(y)}, p_j^{(y)}\} \le B\{k, n^{(x)}, p_i^{(x)}\}\{1 + B[k, n^{(y)}, p_j^{(y)}] - F_B[k, n^{(y)}, p_j^{(y)}]\} + \{1 - F_B[k, n^{(x)}, p_i^{(x)}]\}B\{k, n^{(y)}, p_j^{(y)}\} \quad (35)$$

for $0 \le k \le n^{(x)} \le n^{(y)}$. If $n^{(y)} < n^{(x)}$, interchange $n^{(x)}$ and $n^{(y)}$ in (35). Using (35), the expected number of successful assemblies generated in MTR_r can be computed. Using a normal approximation to the binomial pmf, an approximation for the expected value of U_r for large values of $n^{(x)}$ and $n^{(y)}$ can be found. The expected value of $N_i^{(x)}$ is given by

$$\mu_{X,i} = n^{(x)} p_i^{(x)} \quad (36)$$

and the variance is given by

$$\sigma_{X,i}^2 = n^{(x)} p_i^{(x)}\{1 - p_i^{(x)}\}. \quad (37)$$

Similarly, the expected value of $N_j^{(y)}$ is given by

$$\mu_{Y,j} = n^{(y)} p_j^{(y)} \quad (38)$$

and the variance is given by

$$\sigma_{Y,j}^2 = n^{(y)} p_j^{(y)}\{1 - p_j^{(y)}\}. \quad (39)$$

Let

$$\sigma\{i, j\} = \{\sigma_{X,i}^2 + \sigma_{Y,j}^2\}^{.5}. \quad (40)$$

Then it can be shown that

$$E\{U_r \mid n^{(x)}, p_i^{(x)}, n^{(y)}, p_j^{(y)}\} \cong \frac{\mu_{X,i} + \mu_{Y,j}}{2} + \frac{\exp\{-d/2\}\mu_{X,i}\mu_{Y,j}\{2 - p_i^{(x)} - p_j^{(y)}\}}{\{2\pi\}^{.5}\{\sigma^2(i,j)\}^{1.5}} - \frac{\sigma\{i,j\}}{\{2\pi\}^{.5}} \exp\{-\frac{[\mu_{X,i} - \mu_{Y,j}]^2}{2\sigma^2[i,j]}\} \quad (41)$$

where

$$d = \frac{\sigma^2\{i,j\}\{\mu_{X,i}[1 - p_j^{(y)}] + \mu_{Y,j}[1 - p_i^{(x)}]\} - \mu_{X,i}\mu_{Y,j}\{2 - p_i^{(x)} - p_j^{(y)}\}^2}{\{1 - p_i^{(x)}\}\{1 - p_j^{(y)}\}\sigma^2\{i,j\}}.$$

In the special case that $n^{(x)} = n^{(y)} = m$, of those produced, the fraction of assemblies in MTR_r is given by

$$E\{U_r \mid m, p_i^{(x)}, m, p_j^{(y)}\}/m \cong \frac{p_i^{(x)}+p_j^{(y)}}{2} + \frac{\exp\{-d/2\}\, p_i^{(x)} p_j^{(y)} \{2-p_i^{(x)}-p_j^{(y)}\}}{\{2\pi m\}^{.5}\{p_i^{(x)}[1-p_i^{(x)}]+p_j^{(y)}[1-p_j^{(y)}]\}^{1.5}} -$$

$$\left\{\frac{p_i^{(x)}[1-p_i^{(x)}]+p_j^{(y)}[1-p_j^{(y)}]}{2\pi m}\right\}^{.5} \exp\{-\frac{m[p_i^{(x)}-p_j^{(y)}]^2}{2[p_i^{(x)}(1-p_i^{(x)})+p_j^{(y)}(1-p_j^{(y)})]}\} \qquad (42)$$

where

$$d = m\frac{\{p_i^{(x)}[1-p_i^{(x)}]+p_j^{(y)}[1-p_j^{(y)}]\}\{p_i^{(x)}[1-p_j^{(y)}]+p_j^{(y)}[1-p_i^{(x)}]\}-p_i^{(x)}p_j^{(y)}\{2-p_i^{(x)}-p_j^{(y)}\}^2}{\{1-p_i^{(x)}\}\{1-p_j^{(y)}\}\{p_i^{(x)}[1-p_i^{(x)}]+p_j^{(y)}[1-p_j^{(y)}]\}}.$$

.

It can be seen from (42) that as m increases, the fraction of assemblies in MTR_r converges to the mean of $p_i^{(x)}$ and $p_j^{(y)}$. It can also be shown, using (42), that for $n^{(x)}$ large, as $n^{(y)}$ increases relative to $n^{(x)}$, that

$$E\{U_r \mid n^{(x)}, p_i^{(x)}, n^{(y)}, p_j^{(y)}\}/n^{(x)} \cong \frac{p_i^{(x)}}{2} + \frac{p_j^{(y)} n^{(y)}}{2n^{(x)}}. \qquad (43)$$

7. Discussion on the use of Excel© to compute truncation values

As discussed above, the solver function was used in order to define the four corners of each TR such that the value for the sums Σ_M or Σ_A was a maximum. In the examples, results using Σ_M were shown, but none using Σ_A. The solution that solver obtains can be dependent on the initial choice of values for the variables. It was found that after a solution using Σ_M was obtained, another solution using Σ_A was found, and this solution was then in place when solver was used again for Σ_M. This procedure sometimes yielded a better solution. Another important feature that was used was the 'chart wizard'. The figures were all generated automatically by expressing each corner of every TR as a function of the solution. It was then possible to see the effects of changing the values of the variables, as the figure immediately showed the boundaries of the new TRs. New values for each P_M were also automatically recomputed. These features can help the user to decide what the constraints should be when increasing the number of TRs.

8. Summary

The geometric approach was presented first as a means to introduce the concepts to be used in a simple manner. The uniform pdfs also have the advantage of simplicity of interpretation, but have the disadvantage of needing to generate complex solutions. The solutions generated when using normal pdfs are considered to be typical of what would be encountered in applications, and they are simpler due to the fact the distributions are unimodal. For constrained problems, the shape of the region that lies between the constraining boundaries makes it difficult to find an optimal solution. The use of the 'solver' and 'chart wizard' functions that are contained in Microsoft Excel© are considered to be very helpful in generating solutions that were considered to be near optimal for a specified lower bound on the sum of the probabilities of a sample lying in each group.

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